Lecture 3
Producer theory

Abbreviation	Name	Description
L	Commodities	i = 1,…,L
y	single production plan	$y = (y_1,\ldots,y_L)$
Y	production set	Set of all feasible production plans
F(y)	transformation function	It implicitly defines the frontier of Y, but tells us neither the form of the waste nor the magnitude.
y(p)	net supply function	Supply function for production-plan model
q	single output	
q(w,p)	supply function	
f(z)	production function	The quantity of output produced when input vector *z* is employed by the firm
$f_i(z)$	marginal product	The amount by which output increases if you increase input z_i by a small amount
$pf_i(z)$	marginal revenue product	The amount by which revenue increases if you increase input z_i by a small amount
z(w,p)	firm's factor demand function	How much of the inputs are used at prices *p* and *w*
z(w,q)	firm's conditional factor demand function	How much of the inputs are used at prices *p* conditioned on the level of output
π(p)	profit function	The firm's maximum profit given the prices of inputs and outputs.
c(w,q)	cost function	

Basic properties of all production sets (Y):

Property	Description
nonempty	If Y is empty, then we have nothing to talk about
closed	A set is closed, if it contains its boundary.
no free lunch	If there were a "free lunch," then the firm could make infinite profit just by replicating the free lunch point over and over, which makes the firm's profit maximization problem impossible to solve.
free disposal	The firm can always throw away inputs, if it wants. ⇒ While you have to buy the commodities you are throwing away, you don't have to pay anybody to dispose of it for you.
Irreversibility	The production process cannot be undone.
Possibility of inaction	The firm can choose to do nothing, i.e. $0 \in Y$. Situations where $0 \notin Y$ arise when the firm has a fixed factor of production. Example: If the firm is obligated to pay rent on its factory, then it cannot do nothing.

Returns to scale properties of production sets (Y):

Property	Description
Nonincreasing returns to scale	Any feasible production plan $y \in Y$ can be scaled down: $ay \in Y$ for $a \in [0, 1]$. The production frontier is locally concave and Y is convex.
Nondecreasing returns to scale	Any feasible production plan $y \in Y$ can be scaled up: $ay \in Y$ for $a \geq 1$. The production frontier is locally convex and Y is not convex. • Note that if a firm has fixed costs, it may exhibit nondecreasing returns to scale but cannot exhibit nonincreasing returns to scale.
Constant returns to scale	For all $a \geq 0$, if $y \in Y$, then $ay \in Y$.
Decreasing returns to scale	Returns are nonincreasing and not constant ⇒ The production frontier is locally strictly concave and Y is strictly convex
Increasing returns to scale	Returns are nondecreasing and not constant ⇒ The production frontier is locally strictly convex and Y is strictly concave

Example of typical technology: A manufacturing firm whose factory size is fixed. At first, as output increases, its average productivity increases as it spreads the factory cost over more output. However, eventually the firm's output becomes larger than the factory is designed for. At this point, the firm's average productivity falls as the workers become crowded, machines become overworked, etc.

Properties for the convex preferences

Supply f.	Profit f.	Conditional factor demand f.	Cost f.
Homogeneity of degree zero in p	Homogeneity of degree one in p	Homogeneity of degree zero in w	Homogeneity of degree one in w
If Y is convex ⇒ y(p) is a convex set	Convex (i.e. If the price of an output increases or price of an input decreases and the firm does not change its production plan, profit will increase linearly. However, since the firm will want to re-optimize at the new prices, it can actually do better. Profit will increase at a greater than linear rate.)	If $z \geq 0$ is convex ⇒ z(w,q) is a convex set	Non-decreasing in q
If Y is strictly convex ⇒ y(p) is single point			Concave in w

Important identities

Identities	Description
q=f(z)	Single output function
q(w,p)=f(z(w,p))	
$MRT_{ji} = (\partial F(y)/ \partial y_j)/ (\partial F(y)/ \partial y_i)$	How much you must increase the usage of factor *j* if you decrease the usage of factor *i* in order to remain on the transformation frontier.
$\partial y_j/\partial y_i = - p_i/p_j = -MRT$	The slope of the transformation frontier is negative
$MRTS_{ji} = (\partial f(z)/ \partial z_j)/(\partial f(z)/\partial z_i)$	The amount by which input *j* should be decreased in order to keep output (q=f(z)) constant following an increase in input *i*.
$\partial z_j/\partial z_i = - w_i/w_j = -MRTS$	The slope of the isoquant is negative
Hotelling's lemma $\frac{\partial \pi(p)}{\partial p_i} = y_i(p)$	If y(p) is single-valued at p, then the increase in profit due to an increase in p_i is simply equal to the usage of commodity i.

Profit maximization problem (PMP)[1]

Production-plan (net-output) version of PMP, i.e. we start from purely technological point of view)

At the profit maximizing production plan y*, the marginal rate of transformation between any two commodities is equal to the ratio of their prices. ⇒ The solution of PMP is net supply function y(p).

Note that

- unless Y is convex, the first-order conditions will not be sufficient for a maximum.
- generally, second-order conditions will need to be checked. The second order condition will not hold for increasing returns. ⇒ We will focus on technologies that exhibit nonincreasing returns.
- if Y is convex, Y and π (p) contain the exact same information. A π (p) is analytically much easier to work with than Y

The production-plan approach is necessary to deal with situations where you don't know which commodities will be inputs and which will be outputs. This approach is widely used in the study of general equilibrium, where the output of one industry is the input of another.

[1] The PMP is more similar to EMP that the UMP because both the PMP and EMP do not have to worry abput wealth effects. We may rewrite EMP as $\mathbf{min_{x\geq 0}\{-px \mid u(x) \geq u\}}$

Single-output version of PMP

It is an important special case where the firm produces a single output (q) using a number of inputs:

$$\mathbf{max}_{z\geq 0}\{\mathbf{pq\text{-}wz} \mid \mathbf{f(z)} \geq \mathbf{q}\}$$

Unconstrained maximization problem:

$$\max_{z\geq 0}\{pf(z)\text{-}wz\}$$

For unconstrained maximization problem, we don't need to set up a Lagrangean, but we do need to be concerned with corner solution since the firm may not use all inputs in production.

The Kuhn-Tucker first order conditions:

$$\frac{\partial L}{\partial z_i} = p\frac{\partial f(z^*)}{\partial z_i} - w_i = pf_i(z^*) - w_i \leq 0 \text{ for goods } i = 1,\ldots,L$$

If a firm maximizes profit by producing $q*$ using $z*$, then $z*$ is also the input bundle that produces $q*$ at minimum cost. Consider CMP.

Cost minimization problem (CMP)

Single-output version of CMP

The firm chooses z^* so as to set the marginal rate of technical substitution between any two inputs equal to the ratio of their prices.

The CMP may allow us to say things about firms when the PMP does not:

- The CMP is perfectly well-defined as long as the production function is quasiconcave, even if it exhibits increasing returns. The PMP is not very useful for technologies with increasing returns to scale.
- It is easy to study CMP if the firm is not price-taker on the output market because the cost function does not depends directly from p

Results: Any concept under convexity conditions defined in terms of the properties of the production function

$$\max_{y\geq 0}\{pq|\ c(w,q) \leq c\}$$

has dual definition in terms of the properties of the cost function

$$\min_{z\geq 0}\{wz \mid f(z) \geq q\}$$

and vice versa.